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A Global Existence Theorem for the Nonlinear BGK Equation

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A global existence theorem with large initial data in L^1 is given for the nonlinear BGK equation. The method, which is based on the recent averaging lemma of Golse *et al.*, utilizes a weak compactness argument in L^1.

KEY WORDS: BGK equation; kinetic equation; kinetic theory; semilinear equation.

1. INTRODUCTION

The BGK collision model was proposed in 1954 by Bhatnagar, Gross and Krook (BGK)[1] and independently by Welander[(2)] in the same year. It replaces a large amount of the two-body collision details with some qualitative and average properties of the original Boltzmann collision operator. The BGK equation has played, and still plays, an important role in kinetic theory (see, for example, ref. 7). Until the result presented herein, however, nothing had been known about existence of solutions to (1.1), neither local nor global, not even a near-equilibrium result.

In 1982 we,[(3)] using new ideas on semilinear evolution equations in weak topologies of abstract Banach spaces, obtained a sequence of approximate solutions to (1.1) and its convergence in the weak topology of $L^1(\Omega \times \mathbb{R}^3)$. We were at that time, however, unable to say in what sense the limit of the approximate solutions satisfied the original equation. In this paper a global existence theorem for the nonlinear BGK equation will be proved with initial data in L^1.

Let $f\colon \Omega \times \mathbb{R}^3 \to \mathbb{R}_+ \cup \{0\}$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain

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with smooth boundary, $x \in \Omega$ is the spatial variable, $\xi \in \mathbb{R}^3$ the velocity variable, and $f(x, \xi)$ represents the density distribution at the point x with velocity ξ. Denote by D_0 the set $D_0 = \{f \in L^1(\Omega \times \mathbb{R}^3): f \geqslant 0, (1+\xi^2) f \in L^1(\Omega \times \mathbb{R}^3)\}$, and let $\rho(x) = \int_{\mathbb{R}^3} f(x, \xi)\, d\xi$ be the macroscopic density, $v(x) = \int_{\mathbb{R}^3} \xi f(x, \xi)\, d\xi / \rho(x)$ the macroscopic velocity, $E(x) = \int_{\mathbb{R}^3} \xi^2 f(x, \xi)\, d\xi / 2\rho(x)$ the total macroscopic energy per unit mass, and $T(x) = [2E(x) - v(x)^2]/3R$ the macroscopic temperature, where R is the Boltzmann constant. Define pointwise the BGK collision operator $J(f)(x, \xi) = \nu[P(f)(x, \xi) - f(x, \xi)]$ for

$$P(f)(x, \xi) = \frac{\rho(x)}{[2\pi RT(x)]^{3/2}} \exp\left\{\frac{-[\xi - v(x)]^2}{2RT(x)}\right\}$$

The BGK equation can be written as

$$\frac{\partial f}{\partial t} = -\xi \cdot \nabla_x f + J(f), \qquad f(0, x, \xi) = f_0(x, \xi) \tag{1.1}$$

for $t \in \mathbb{R}_+$, $x \in \Omega$, $\xi \in \mathbb{R}^3$.

The BGK collision operator describes a gas tending to a Maxwellian distribution. Indeed, the average effect of collisions changes $f(t, x, \xi)$ by an amount proportional to its departure from a local Maxwellian P. The qualitative features of the Boltzmann collision operator—conservation of mass, momentum, and energy—are expressed by the equalities

$$\int_{\mathbb{R}^3} \psi_i J(f)\, d\xi = 0, \qquad \text{a.e. in } x \tag{1.2}$$

for $i = 0, 1, 2, 3, 4$ and $f \in D_0$, where ψ_i are the collision invariants $\psi_0 = 1$, $\psi_i = \xi_i$ for $i = 1, 2, 3$ (components of ξ), and $\psi_4 = \xi^2$. Formally, at least, we have also the Boltzmann inequality

$$\int_{\mathbb{R}^3} J(f) \log f\, d\xi \leqslant 0 \tag{1.3}$$

for fixed x, with equality if and only if $f = P(f)$.

The collision frequency ν can be a function of the local state of the gas, i.e., $\nu(\rho, v, E)$, thus depending on x and t. In spite of this, for reasons of simplicity, ν has nearly always been taken to be a constant. For the results to be given here, however, it will be crucial that ν is a function of the local state of the gas.

Recently, DiPerna and Lions,[4] leaning on a new compactness argument due to Golse *et al.*,[5] have proved a global existence theorem for the nonlinear Boltzmann equation. Here, also, we shall exploit the velocity averaging lemma of ref. 5, along with properties of the BGK collision operator obtained in ref. 3, to obtain global existence for the nonlinear BGK equation.

2. BASIC PROPERTIES AND APPROXIMATE SOLUTIONS OF THE BGK EQUATION

We start with a few definitions that set up the BGK equation (1.1) in the framework of semilinear evolution equations. Let

$$L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3) = \left\{ f \in L^1(\Omega \times \mathbb{R}^3): \quad \|f\|_2 \equiv \iint_{\Omega \times \mathbb{R}^3} (1+\xi^2)\, |f|\, d\xi\, dx < \infty \right\}$$

Here Ω is a three-dimensional torus, i.e., $\Omega = \mathbb{R}^3/\mathbb{Z}^3$. The choice of Ω is a convenient way to express the fact that we consider the operator $Af \equiv -\xi \cdot \nabla_x f$ with periodic boundary conditions. It is well known (see, for example, ref. 6) that A generates a strongly continuous semigroup $T(t)$ in $L^1(\Omega \times \mathbb{R}^3)$, and that the restriction of $T(t)$ to $L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$ [also denoted by $T(t)$] is a strongly continuous semigroup in $L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$. For $M > 0$ and $C \in \mathbb{R}$ we define the set D by

$$D = \{ f \in L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3): \quad f \geqslant 0,\ \|f\|_2 \leqslant M,\ H(f) \leqslant C \}$$

where the functional H is given by

$$H(f) = \iint_{\Omega \times \mathbb{R}^3} f \log f\, d\xi\, dx$$

Because of the conservation laws, the set D is a natural domain for the BGK collision operator.

We say that a continuous function f from $[0, a]$, $a > 0$, into $D \subset L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$ is a mild solution to (1.1) in $L_{1+\xi^2}(\Omega \times \mathbb{R}^3)$ if it satisfies

$$f(t) = T(f)\, f_0 + \int_0^t T(t-s)\, J(f(s))\, ds \tag{2.1}$$

for $t \in [0, a]$. The sense of integration is taken generally in the Bochner sense. It will, in fact, turn out to be the Riemann integral in $L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$ for the problem of interest here.

We give next the velocity averaging lemma of ref. 5.

Lemma 2.1. Suppose that $f_n \in L^1((0,a)\times\mathbb{R}^3\times\mathbb{R}^3)$ and $g_n \in L^1((0,a)\times\mathbb{R}^3\times\mathbb{R}^3)$ satisfy

$$T_\xi f_n \stackrel{\text{def}}{=} \frac{\partial f_n}{\partial t} + \xi\cdot\nabla_x f_n = g_n \tag{2.2}$$

in $\mathscr{D}'((0,a)\times\mathbb{R}^3\times\mathbb{R}^3)$, and the sequences $\{f_n\}$ and $\{g_n\}$ are relatively weakly compact in $L^1((0,a)\times\mathbb{R}^3\times\mathbb{R}^3)$. Then for all $\varphi\in L^\infty((0,a)\times\mathbb{R}^3\times\mathbb{R}^3)$ the set $\{\int_{\mathbb{R}^3}\varphi f_n\,d\xi\} = \{\int_{\mathbb{R}^3}\varphi T_\xi^{-1} g_n\,d\xi\}$ is relatively compact in $L^1((0,a)\times\mathbb{R}^3)$.

In other words, the velocity-averaged operator T_ξ^{-1} behaves in a manner similar to the inverse of an elliptic operator. We recall that T_ξ^{-1} may be singular only on the set of the characteristic direction. Velocity averaging compensates for the lack of regularity in the characteristic direction of the hyperbolic operator.

Next, we state a proposition and a theorem presented in ref. 3 under the assumption that the collision frequency ν is a constant. However, the results in ref. 3 extend in a straightforward way to the case $\nu\in L^\infty(\Gamma)$ for $\Gamma\equiv\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+$. The point is that the set $\{f\cdot g: f\in B, g\in C\}$ is a relatively weakly compact set of $L^1(\Omega\times\mathbb{R}^3)$ if B is a bounded subset of $L^\infty(\Omega\times\mathbb{R}^3)$ and C is a relatively weakly compact set of $L^1(\Omega\times\mathbb{R}^3)$.

The proposition follows from the lower semicontinuity of the convex functional H in L^1, together with the Dunford–Pettis theorem and the conservation laws (1.2).

Proposition 2.2. D is a convex weakly compact subset of $L^1(\Omega\times\mathbb{R}^3)$, invariant under the semigroup $T(t)$, $t>0$, and for each sequence $\{f_n\}\subset D$ there exists a subsequence $\{f_{n_i}\}$ and $f\in D$ such that

$$\iint_{\Omega\times\mathbb{R}^3}\varphi f_{n_i}\,d\xi\,dx \xrightarrow[i\to\infty]{} \iint_{\Omega\times\mathbb{R}^3}\varphi f\,d\xi\,dx$$

for all measurable φ satisfying $|\varphi(x,\xi)|\leqslant(1+\xi^2)^k$, $k<1$. Moreover, $P(D)\subseteq D$, and P is continuous as a map from $D\subset L^1_{1+\xi^2}(\Omega\times\mathbb{R}^3)$ into $L^1_{1+\xi^2}(\Omega\times\mathbb{R}^3)$.

As a consequence of the proposition, the following theorem is the main result of ref. 3.

Theorem 2.3. For each $f_0\in D$ and each sequence $\{\varepsilon_n\}$ with $0<\varepsilon_n\to 0$ there exists a sequence $\{f_n\}$ of approximate solutions to (2.1); i.e., for each $n\geqslant 1$ there exists $\{t_i^n\}_{i=1}^{N(n)}\subset[0,a]$ with $t_0^n=0$, $t_{i+1}^n - t_i^n\leqslant\varepsilon_n$ for $i=1,2,\ldots,N(n)$, and $t_{N(n)}^n=a$ such that $f_n(0)=f_0$, $f_n(t_i^n)\in D$, and

(a) $f_n(t) = T(t - t_i^n) f_n(t_i^n) + \int_{t_i^n}^t T(t-s) J(f_n(t_i^n))\, ds$ for $t \in [t_i^n, t_{i+1}^n)$

(b) $\|f_n(t_{i+1}^n) - f_n(t_i^n)\|_2 \leqslant \varepsilon_n (t_{i+1}^n - t_i^n)$

(c) $\|f_n(t) - T(t) f_0 - \int_0^t T(t-s) J[f_n(\gamma_n(s))]\, ds\|_2 \leqslant t_i^n \varepsilon_n$
for $t \in [t_i^n, t_{i+1}^n)$, where $\gamma_n(t) = t_i^n$ if $t \in [t_i^n, t_{i+1}^n)$ and $\gamma_n(a) = a$

Furthermore, $\{f_n\}$ contains a subsequence $\{f_{n_i}\}$ which converges weakly in $L^1(\Omega \times \mathbb{R}^3)$ and uniformly on $[0, a]$ to a limit f, where f: $[0, a] \to D$ is weakly continuous in $L^1(\Omega \times \mathbb{R}^3)$.

We note that the integrals above are Riemann integrals in $L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$. Theorem 2.3 is valid for any bounded $\Omega \subset \mathbb{R}^3$ with any boundary conditions that guarantee $A: f \to -\xi \cdot \nabla_x f$ to be the generator of a strongly continuous semigroup $T(t)$ satisfying $T(t) D \subseteq D$ (for example, stochastic boundary conditions). The choice of Ω in this paper is motivated by the fact that the velocity averaging lemma is presently known for $\Omega = \mathbb{R}^3$ or for periodic boundary conditions.

The collision frequency $\nu = \nu(\rho, v, E)$ will be taken throughout as an essentially bounded measurable function of the normalized moments. We will need one additional assumption. We shall say that the collision frequency satisfies the *energy saturation condition* if $\nu(\rho, v, E) \to_{E \to \infty} 0$ uniformly in ρ and v. An example of energy saturation is provided by ν with compact support with respect to E. Indeed, since the energy density per unit mass is proportional to $\int \xi^2 f\, d\xi / \int f\, d\xi$, this corresponds to the saturation of the energy density per unit mass in the BGK collision model.

3. PASSING TO THE LIMIT AND THE EXISTENCE THEOREM

We start with the following preliminary result.

Proposition 3.1. Let $\{f_n\}$ be a sequence of approximate solutions to (2.1) obtained in Theorem 2.3. Then there exists a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ and a weakly continuous function f from $[0, a]$ into $D \subset L^1(\Omega \times \mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \psi f_{n_i}\, d\xi \xrightarrow[i \to \infty]{} \int_{\mathbb{R}^3} \psi f\, d\xi$$

a.e. in t and x and for all measurable ψ with $|\psi(x, \xi)| \leqslant c(1 + \xi^2)^k$, $k < 1$.

Proof. By Theorem 2.3, after passing to a subsequence if necessary, $f_n(t) \to_{n \to \infty} f(t)$ weakly in $L^1(\Omega \times \mathbb{R}^3)$, uniformly in t, where f: $[0, a] \to D \subset L^1(\Omega \times \mathbb{R}^3)$ is weakly continuous.

Observe that

$$h_n(t) = T(t) f_0 + \int_0^t T(t-s) J(f_n(\gamma_n(s))) ds$$

is a solution in $\mathscr{D}'((0, a) \times \Omega \times \mathbb{R}^3)$ to (2.2) with $g_n = J(f_n)$. By Theorem 2.3 and Proposition 2.2, $\{h_n\}$ and $\{g_n\}$ are relatively weakly compact sets in $L^1((0, a) \times \Omega \times \mathbb{R}^3)$. Thus, by Lemma 2.1, for each $\psi \in L^\infty((0, a) \times \Omega \times \mathbb{R}^3)$ the set $\{\int_{\mathbb{R}^3} \psi h_n \, d\xi\}$ is relatively compact in $L^1((0, a) \times \Omega)$. Also, by using property (c) of the approximate solution and the fact that $\varepsilon_n \to 0$, the set $\{\int_{\mathbb{R}^3} \psi f_n \, d\xi\}$ has compact closure in $L^1((0, T) \times \Omega)$. Finally, since $\|f_n\|_2 \leqslant M < \infty$ for $n \geqslant 1$, this last assertion is also true for each measurable ψ with $|\psi(x, \xi)| \leqslant c(1 + \xi^2)^k$, $k < 1$. This completes the proof. ∎

It is important to notice that one cannot show $\int_{\mathbb{R}^3} \xi^2 f_{n_i} \, d\xi \to \int_{\mathbb{R}^3} \xi^2 f \, d\xi$ a.e. in t and x directly from the velocity averaging lemma. This has significant consequences, in particular, for the Boltzmann equation, and is reflected by the failure of the DiPerna–Lions proof to demonstrate conservation of energy. In fact, their proof fails for scattering potentials with $B(\theta, |V|) \approx |V|^2$, i.e., with the second moment explicitly built into the collision operator as is the case for the BGK equation.

Proposition 3.2. Let $\{f_n\}$ be a sequence of approximate solutions to (2.1) obtained in Theorem 2.3. If the collision frequence ν satisfies the energy saturation condition, then for some subsequence $\{f_{n_i}\}$ of $\{f_n\}$, $\int_{\mathbb{R}^3} \xi^2 f_{n_i} \, d\xi$ converges a.e. in t and x.

Proof. First, we observe that, by Proposition 3.1, for each $R > 0$ the set $\{\int_{|\xi| \leqslant R} \xi^2 f_n \, d\xi\}$ has compact closure in $L^1((0, a) \times \Omega)$. Therefore, in order to obtain the desired convergence it is enough to show that

$$\int_0^a \int_\Omega \int_{|\xi| \geqslant R} \xi^2 f_n \, d\xi \, dx \, dt \xrightarrow[R \to \infty]{} 0 \quad \text{uniformly in } n \geqslant 1$$

Property (c) of the approximate solution and Gronwall's lemma applied to

$$p_n^R(t) = \int_\Omega \int_{|\xi| \geqslant R} \xi^2 f_n(t, x, \xi) \, d\xi \, dx$$

imply that it is sufficient to show

$$A(n, t, R) = \int_\Omega \int_{|\xi| \geqslant R} \int_0^t \xi^2 T(t-s) \{(\nu P)[f_n(\gamma_n(s))]\} \, ds \, d\xi \, dx \xrightarrow[R \to \infty]{} 0 \tag{3.1}$$

uniformly in $n \geqslant 1$ and $t \in [0, a]$. After easy but tedious integration, we can obtain

$$A(n, t, R) = 4\pi \int_0^t ds \int_{|v_n| \geqslant R} dx\, \nu(\rho_n, v_n, E_n) \Bigg[e_n \rho_n \int_0^{M_n^-} z^{3/2} e^{-z}\, dz$$

$$+ 4v_n (e_n \rho_n)^{1/2} \int_0^{M_n^-} z e^{-z}\, dz + 2v_n^2 \rho_n \int_0^{M_n^-} z^{1/2} e^{-z}\, dz \Bigg]$$

$$+ 4\pi \int_0^t ds \int_\Omega dx\, \nu(\rho_n, v_n, E_n)$$

$$\times \Bigg[e_n \rho_n \int_{M_n^+}^\infty z^{3/2} e^{-z}\, dz + 4v_n (e_n \rho_n)^{1/2} \int_{M_n^+}^\infty z e^{-z}\, dz$$

$$+ 2v_n^2 \rho_n \int_{M_n^+}^\infty z^{1/2} e^{-z}\, dz \Bigg] = I_- + I_+ \tag{3.2}$$

where $\rho_n = \int_{\mathbb{R}^3} f_n\, d\xi$, $v_n \rho_n = \int_{\mathbb{R}^3} \xi f_n\, d\xi$, $2E_n \rho_n = \int_{\mathbb{R}^3} \xi^2 f_n\, d\xi$, $e_n = 2E_n - v_n^2$, and $M_n^\pm = (|v_n| \pm R)^2 / e_n$. Since $E_n \geqslant R/2$ for all $n \geqslant 1$ in the term I_-, the assumption on ν implies that $I_- \to 0$ as $R \to \infty$. The term I_+ can be written as

$$I_+ = 4\pi \int_0^t ds \left\{ \int_{\Delta_1} dx + \int_{\Delta_2} dx + \int_{\Delta_3} dx \right\} \nu(\rho_n, v_n, E_n)[\cdots] = L_1 + L_2 + L_3$$

where $\Delta_1 = \{x\colon |v_n| \geqslant \sqrt{R}\}$, $\Delta_2 = \{x\colon |v_n| \leqslant \sqrt{R}, E_n \geqslant R\}$, and $\Delta_3 = \{x\colon |v_n| \leqslant \sqrt{R}, E_n \leqslant R\}$. The terms L_1 and L_2 approach zero as $R \to \infty$ by the same argument that was applied to I_-. On the other hand, $L_3 \to 0$ as $R \to \infty$ because in this term we have $M_n^+ \geqslant R/2$. ∎

Theorem 3.3. Suppose the collision frequency ν satisfies $\nu \in L^\infty(\Gamma)$ and the energy saturation condition, and $f_0 \in D$. Then there exists a global mild solution to the BGK equation in $L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$.

Proof. Propositions 3.1 and 3.2 imply that for a.e. $s \in [0, a]$ the set $\{J(f_n(\gamma_n(s)))\}$ has compact closure in $L^1(\Omega \times \mathbb{R}^3)$. The strong continuity of $T(t)$ in $L^1(\Omega \times \mathbb{R}^3)$ implies that the set $\{k_n\}$ is relatively compact in $L^1((0, a), L^1(\Omega \times \mathbb{R}^3))$, where

$$k_n(t) = \int_0^t T(t-s)\, J(f_n(\gamma_n(s)))\, ds \qquad \text{for} \quad t \in [0, a]$$

By property (c) of the approximate solution, the set $\{f_n\}$ is relatively compact in $L^1((0, a), L^1(\Omega \times \mathbb{R}^3))$. Therefore, there exists $f \in L^1((0, a),$

$L^1(\Omega \times \mathbb{R}^3))$ with $f(t) \in D$ a.e. in t and such that, after passing to a subsequence, if necessary,

$$\int_{\mathbb{R}^3} \xi^2 f_{n_i}\, d\xi \to \int_{\mathbb{R}^3} \xi^2 f\, d\xi \qquad \text{a.e. in } t \text{ and } x$$

Next, by using the dominated convergence theorem for the Bochner integral in $L^1(\Omega \times \mathbb{R}^3)$, we obtain that in $L^1((0, a), L^1(\Omega \times \mathbb{R}^3))$

$$f(t) = T(t)\, f_0 + \int_0^t T(t-s)\, J(f(s))\, ds \tag{3.3}$$

where the integral is the Bochner integral in $L^1(\Omega \times \mathbb{R}^3)$. This equation implies that f is continuous from $[0, a]$ into $D \subset L^1(\Omega \times \mathbb{R}^3)$. Therefore, by conservation of energy, we conclude that f is continuous from $[0, a]$ into $D \subset L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$. Thus, since J is continuous as a map from $D \subset L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$ into $L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$, f is a mild solution to (2.1) in $L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$, and the integral in (3.3) can be taken as the Riemann integral in $L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$. ∎

Corollary 3.4. The solution obtained in Theorem 3.3 satisfies energy conservation: $\|f(t)\|_2 = \|f_0\|_2$ for $t \in [0, a]$.

Corollary 3.5. If $v \in L^\infty(\Gamma)$ and one of the higher moments of f_n is uniformly bounded, i.e.,

$$\int_0^a \int_\Omega \int_{\mathbb{R}^3} |\xi|^k f_n\, d\xi\, dx\, dt \equiv M_n^k < c < \infty \tag{3.4}$$

for some fixed $k > 2$, then there exists a global mild solution to the BGK equation in $L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$.

Indeed, the boundedness of the sequence $\{M_n^k\}_{n=1}^\infty$ leads to the compactness of the closure of the set $\{\int_{\mathbb{R}^3} \xi^2 f_n\, d\xi\}$ in $L^1((0, a) \times \Omega)$ and consequently, after passing to a subsequence if necessary, to the estimate

$$\int_{\mathbb{R}^3} \xi^2 f_{n_i}\, d\xi \to \int_{\mathbb{R}^3} \xi^2 f\, d\xi \qquad \text{a.e. in } t \text{ and } x$$

The condition in Corollary 3.5 is independent of the energy saturation condition. It is, however, related. Indeed, with the help of the Gronwall lemma, property (c) of the approximate solution, and the boundedness of $\iint_{\Omega \times \mathbb{R}^3} |\xi|^k f_0\, d\xi\, dx$, the sequence $\{M_n^k\}_{n=1}^\infty$ is bounded if the sequence $\{L^k(t, f_n)\}_{n=1}^\infty$ is bounded uniformly in $t \in [0, a]$. Here $L^k(t, f_n)$ is given by

$$L^k(t, f_n) = \int_0^t \int_\Omega \int_{\mathbb{R}^3} |\xi|^k\, T(t-s)(vP)[f_n(\gamma_n(s))]\, d\xi\, dx\, ds$$

An easy integration shows that $L^k(t, f_n)$ is a sum of terms that are equal or bounded by

$$C^k(f_n) = \int_0^a \int_\Omega \frac{(\int_{\mathbb{R}^3} \xi^2 f_n \, d\xi)^{k/2}}{(\int_{\mathbb{R}^3} f_n)^{k/2-1}} \, v \, dx \, ds$$

But, for example, if $v \in L^\infty(\Gamma)$, $k > 2$, and $v(\alpha, \beta, \delta) \approx (\delta^{k/2-1})^{-1}$ for large δ uniformly in α and β, then a bound on $C^k(f_n)$ can be obtained.

In closing, we note that under the conditions of Theorem 3.3, the sequence of approximate solutions $\{f_n\}$ converges strongly in the space $L^\infty((0, a), L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3))$ to a mild solution $f \in C([0, a], L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3))$. This contrasts with the (weak) sense of convergence of the sequence of approximate solutions constructed by DiPerna and Lions in the case of the Boltzmann equation. Actually, their notion of a solution is much weaker than the notion of mild solution in $L^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$ obtained here for the BGK equation. Finally, as noted earlier, energy conservation is not yet known for the DiPerna–Lions solution to the Boltzmann equation.

Note Added in Proof. After the completion of this work, B. Perthame provided in a preprint a short proof that the moment bound (3.4) is always valid for $k = 3$ if it is satisfied by the initial datum $f_0(x, \xi)$. Therefore, according to Corollary 3.5, for such initial conditions the energy saturation condition is not necessary.

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REFERENCES

1. P. L. Bhatnagar, E. P. Gross, and M. Krook, A model for collision processes in gases. I. Small amplitude processes in charged and neutral one-component systems, *Phys. Rev.* **94**:511–525 (1954).
2. P. Welander, On the temperature jump in a rarefied gas, *Ark. Fys.* 7:507 (1954).
3. J. Polewczak and W. Greenberg, Some remarks about continuity properties of local Maxwellians and an existence theorem for the BGK model of the Boltzmann equation, *J. Stat. Phys.* **33**:307–316 (1983).
4. R. L. DiPerna and P. L. Lions, On the Cauchy problem for the Boltzmann equation: Global existence and weak stability, preprint (1988).
5. F. Golse, P. L. Lions, B. Perthame, and R. Sentis, Regularity of the moments of the solution of a transport equation, *J. Funct. Anal.* **76**:110–125 (1988).
6. J. Voigt, The *H*-theorem for Boltzmann type equations, *J. Reine Angew. Math.* **326**:198–213 (1981).
7. C. Cercignani, *Theory and Application of the Boltzmann Equation* (Elsevier, New York, 1975).

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